

FROBENIUS MANIFOLDS FROM SUBREGULAR CLASSICAL W -ALGEBRAS

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ABSTRACT. We obtain algebraic Frobenius manifolds from classical W -algebras associated to subregular nilpotent elements in simple Lie algebras of type D_r where r is even and E_r . The resulting Frobenius manifolds are certain hypersurfaces in the total spaces of semiuniversal deformation of simple hypersurface singularities of the same types.

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1. INTRODUCTION

A **Frobenius manifold** M is a manifold with the structure of Frobenius algebra on the tangent space $T_z M$ at any point $z \in M$ with a flat invariant bilinear form (\cdot, \cdot) and an identity e plus some compatibility conditions [13]. We say M is semisimple or massive if $T_z M$ is semisimple for generic z . Locally, in the flat coordinates (t^1, \dots, t^r) , the structure of Frobenius manifold is encoded in a potential $F(t^1, \dots, t^r)$ satisfying a system of partial differential equations known in topological field theory as the Witten-Dijkgraaf-Verlinde-Verlinde (WDVV) equations. We consider Frobenius manifolds where the quasihomogeneity condition takes the form

$$\sum_{i=1}^r d_i t_i \partial_{t_i} F(t) = (3 - d) F(t) \quad (1.1)$$

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where $e = \partial_{t^{r-1}}$ and $d_{r-1} = 1$. This condition defines **the degrees** d_i and **the charge** d of M . If $\mathbb{F}(t)$ is an algebraic function we call M an **algebraic Frobenius manifold**. **Dubrovin conjecture** on classification of algebraic Frobenius manifolds is stated as follows: semisimple irreducible algebraic Frobenius manifolds with positive degrees d_i correspond to quasi-Coxeter (primitive) conjugacy classes in irreducible Coxeter groups. A quasi-Coxeter conjugacy class in an irreducible Coxeter group is a Conjugacy class which has no representative in a proper Coxeter subgroup [3].

There are two major results support the conjecture. First, the conjecture arises from studying the algebraic solutions to associated equations of isomonodromic deformation of algebraic Frobenius manifolds [13],[16]. It leads to quasi-Coxeter conjugacy classes in Coxeter groups by considering the classification of finite orbits of the braid group action on tuple of reflections obtained in [28]. Therefore, it remains the problem of constructing all these algebraic Frobenius manifolds. Second, Dubrovin constructed polynomial Frobenius structures on the orbit spaces of Coxeter groups [12] using the work of [24]. Then Hertling [17] proved that these are all possible **polynomial Frobenius manifolds**. The isomonodromic deformation of Polynomial Frobenius manifolds lead to Coxeter conjugacy classes [13].

The classification of polynomial Frobenius manifolds reveals a relation between the order and eigenvalues of the conjugacy class, and the charge and degrees of the corresponding Frobenius manifold. More precisely, If the order of the conjugacy class is $\kappa + 1$ and the eigenvalues are $\exp \frac{2\eta_i \pi i}{\kappa+1}$ then the charge of the Frobenius manifold is $\frac{\kappa-1}{\kappa+1}$ and the degrees are $\frac{\eta_i+1}{\kappa+1}$. We depend on this **weak relation** in considering a new examples of algebraic Frobenius manifolds.

In [8] we continue the work of [23] and we began to develop a construction of algebraic Frobenius manifolds using Drinfeld-Sokolov reduction. This means we restrict ourself to conjugacy classes in Weyl groups. The examples obtained correspond, in the notations of [3], to the conjugacy classes $D_4(a_1)$ and $F_4(a_2)$. In [10] we succeeded to uniform the construction of all polynomial Frobenius manifolds. In this work we uniform the construction of algebraic Frobenius manifolds which correspond to the conjugacy classes $D_r(a_1)$ where r is even and $E_r(a_1)$.

In order to formulate the main results of this work, let us recall the relation between subregular nilpotent elements and deformation of simple hypersurface singularities [26]. Let \mathfrak{g} be a simple Lie algebra of type D_r where r is even or E_r . Fix a **subregular nilpotent element** e in \mathfrak{g} . By definition a nilpotent element is called subregular if its centralizer in \mathfrak{g} is of dimension $r + 2$. We fix, by using the Jacobson-Morozov theorem, a semisimple element h and a nilpotent element f such that $\mathcal{A} = \{e, h, f\}$ is an sl_2 -triple, i.e

$$[h, e] = 2e; [h, f] = -2f; [e, f] = h. \quad (1.2)$$

The action of \mathcal{A} decompose \mathfrak{g} to $r + 2$ irreducible \mathcal{A} -submodules

$$\mathfrak{g} = \bigoplus_{i=1}^{r+2} V^i \quad (1.3)$$

Let $\dim V^i = 2\eta_i + 1$. We call the set

$$Wt(e) = \{\eta_i : i = 1, \dots, r + 2\} \quad (1.4)$$

the weights of e . Let $\eta_0 + 1$ be the Coxeter number of \mathfrak{g} . The set $Wt(e)$, under our choice of a total order, together with η_0 are given in the following table.

\mathfrak{g}	$Wt(e)$			
	η_0	$\eta_1, \dots, \eta_{r-1}$	η_r	η_{r+1}, η_{r+2}
D_r	$2r-3$	$1, 3, \dots, r-1; r-1, r+1, \dots, 2r-5$	r-3	$1, r-2$
E_6	11	$1, 4, 5, 7, 8$	2	$3, 5$
E_7	17	$1, 5, 7, 9, 11, 13$	3	$5, 8$
E_8	29	$1, 7, 11, 13, 17, 19, 23$	5	$9, 14$
	$Et(\mathfrak{g})$			

We observe that the set $Et(\mathfrak{g})$ of the exponents of \mathfrak{g} is

$$Et(\mathfrak{g}) = \{\eta_i : i = 0, \dots, r-1\}. \quad (1.5)$$

Let G be the adjoint group of \mathfrak{g} . By Chevalley theorem, the algebra $S(\mathfrak{g}^*)^G$ of invariant polynomials under the adjoint action of G is generated by r homogenous polynomials of degrees $\eta_i + 1$, $i = 0, \dots, r-1$. We fix a homogenous generator $\chi^0, \dots, \chi^{r-1}$ of the algebra $S(\mathfrak{g}^*)^G$ with degree χ^i equals $\eta_i + 1$. Then let us consider the **adjoint quotient** map

$$\chi = (\chi^0, \dots, \chi^{r-1}) : \mathfrak{g} \rightarrow \mathbb{C}^r \quad (1.6)$$

and define the **Slodowy slice**

$$Q = e + \mathfrak{g}^f; \quad \mathfrak{g}^f := \ker \text{ad } f. \quad (1.7)$$

Then Brieskorn proved that the restriction of χ to Q is semiuniversal deformation of the simple hypersurface singularity $\mathcal{N} \cap Q$ [2] which is of the same type as \mathfrak{g} .

Let us introduce the following coordinates on Q

$$\sum_{i=1}^{r+2} z^i X_{-\eta_i}^i + e \in Q$$

where $X_{-\eta_i}^i$ is a normalized minimal weight vector of V^i . We assign the degree $2\eta_i + 2$ to z^i . Then a simple modification of the work in [25] we get the following

Proposition 1.1. *The map $\chi|_Q$ has rank $r-1$ at e . We can normalize the modules V^i and choose a homogenous generators $\chi^0, \dots, \chi^{r-1}$ for $S(\mathfrak{g}^*)^G$ such that the restriction t^i of χ^i to Q with $i > 0$ take the form*

$$t^i = z^i + \text{non linear terms}. \quad (1.8)$$

In particular, setting

$$t^{r+i} = z^{r+i}, \quad i = 0, 1, 2 \quad (1.9)$$

we get a quasihomogenous coordinates (t^1, \dots, t^{r+2}) on Q with degree t^i equals degree z^i .

We will call the coordinates (t^1, \dots, t^{r+2}) on Q obtained in this proposition **Slodowy coordinates**. In this coordinates the restriction of the quotient map to Q take the form

$$\chi|_Q : (t^1, \dots, t^{r+2}) \mapsto (t^0, t^1, \dots, t^{r-1}) \quad (1.10)$$

where t^0 is the restriction to Q of the invariant polynomial χ^0 .

Let us consider the Lie-Poisson bracket P on \mathfrak{g} . The Dirac reduction of P to Q give a nontrivial Poisson bracket $\{.,.\}^Q$. It is know in the literature as the adjoint transverse Poisson structure to the nilpotent orbit of e . In [5] they prove the following

Theorem 1.2. *The matrix $F^{ij}(t) = \{t^i, t^j\}^Q$ is constant multiple of the matrix*

$$\begin{pmatrix} 0 & 0 \\ 0 & \Omega \end{pmatrix} \quad (1.11)$$

where Ω is a 3×3 matrix of the form

$$\begin{pmatrix} 0 & \frac{\partial t^0}{\partial t^{r+2}} & -\frac{\partial t^0}{\partial t^{r+1}} \\ -\frac{\partial t^0}{\partial t^{r+2}} & 0 & \frac{\partial t^0}{\partial t^r} \\ \frac{\partial t^0}{\partial t^{r+1}} & -\frac{\partial t^0}{\partial t^r} & 0 \end{pmatrix}. \quad (1.12)$$

Let $N \subset Q$ be the hypersurface of dimension r defined as follows

$$N = \left\{ t \in Q : \frac{\partial t^0}{\partial t^{r+2}} = \frac{\partial t^0}{\partial t^{r+1}} = 0 \right\} \quad (1.13)$$

It will follow that $\frac{\partial t^0}{\partial t^{r+2}}$ depends linearly on t^{r+2} and $\frac{\partial t^0}{\partial t^{r+1}}$ is a polynomial in t^{r+1} of degree $r-2$ (resp. 2) if \mathfrak{g} is a Lie algebra of type D_r (resp. E_r). In particular, (t^1, \dots, t^r) are well defined coordinates on N . Let $\kappa = \max Wt(e)$. Then we proved the following

Theorem 1.3. *The space N has a natural structure of algebraic Frobenius manifold with charge $\frac{\kappa-1}{\kappa+1}$ and degrees $\frac{\eta_i+1}{\kappa+1}$, $i = 1, \dots, r$.*

By natural we mean that it can be formulated entirely in terms of the representation theory of \mathcal{A} . The potential \mathbb{F} of this Frobenius structure depends on the solution of the equation $\frac{\partial t^0}{\partial t^{r+1}} = 0$. The set

$$Et(e) = \{\eta_i : i = 1, \dots, r\} \quad (1.14)$$

plays the same role as the set $Et(\mathfrak{g})$ for polynomial Frobenius manifolds and we call it **the exponents of e** . Here is some brief details about what we did in order to prove the theorem above.

- (1) We review the relation between the nilpotent element and the conjugacy class. Using the work of [27] and [7], we fix a number ρ such that $y_1 = e + \rho X_{-\eta_\kappa}^\kappa$ is regular semisimple. Then $\mathfrak{h}' = \ker y_1$ is Cartan subalgebra and it is known as **the opposite Cartan subalgebra**. The element $w := \exp \frac{2\pi i}{\kappa+1} \text{ad } h$, acts on \mathfrak{h}' as a representative of the conjugacy class $D_r(a_1)$ (resp. $E_r(a_1)$) if \mathfrak{g} is of type D_r (resp. E_r).
- (2) We consider the standard local Poisson bracket on the loop algebra $\mathfrak{L}(\mathfrak{g})$. Then we use the subalgebra \mathcal{A} to perform Drinfeld-Sokolov reduction ([8],[19],[1]) or equivalently the Dirac reduction [19],[9] to get a local Poisson structure $\{.,.\}^{\tilde{Q}}$ on Slodowy slice

$$\tilde{Q} = e + \mathfrak{L}(\mathfrak{g}^f); .$$

This Poisson bracket is known in the literature as the **classical W -algebras** associated to the nilpotent orbit of e [19]. The leading term of this Poisson bracket is the adjoint transverse Poisson structure $\{.,.\}^Q$.

- (3) we perform the Dirac reduction on $\{.,.\}^{\tilde{Q}}$ in order to obtain a local Poisson structure which admits a dispersionless limit. This is possible on a loop space $\tilde{N} := \mathfrak{L}(N)$ of the hypersurface N defined above. The new local Poisson bracket $\{.,.\}^{\tilde{N}}$ is also a classical W -algebra. We call it **subregular classical W -algebra**. Then from $\{.,.\}^{\tilde{N}}$ we get a local Poisson bracket of hydrodynamic type $\{.,.\}^{[0]}$. In the coordinates (t^1, \dots, t^r) of N we have by definition

$$\{t^i(x), t^j(y)\}^{[0]} = g^{ij}(t(x))\delta'(x-y) + \Gamma_k^{ij}(t(x))t_x^k\delta(x-y). \quad (1.15)$$

where $g^{ij}(t(x))$ and $\Gamma_k^{ij}(t(x))$ are polynomials in $t^i(x)$.

- (4) We transfer to finite dimensional geometry. We check that the matrix $g^{ij}(t(x))$ is nondegenerate. It follows that the nondegeneracy condition could be traced back to the existence of opposite Cartan subalgebra \mathfrak{h}' . Then from Dubrovin-Novikov theorem, $g^{ij}(t)$ define a flat contravariant metric on N . Moreover, from the structure of the matrix $g^{ij}(t)$, it follows that the matrix $\partial_{t^{r-1}}g^{ij}$ define another flat contravariant metric on N . Furthermore, the two metrics form a flat pencil of metrics on N .
- (5) We prove that the flat pencil of metrics on N satisfies the quasihomogeneity and the regularity conditions. The proof depends on the definition of classical W -algebra and the structure of the set $Et(e)$. Then we obtain a Frobenius structure on N using a theorem and construction due to Dubrovin [14]. The resulting Frobenius manifold satisfy the weak relation.

2. PRELIMINARIES

2.1. Flat pencil of metrics and Frobenius manifolds. In this section we review the relation between the geometry of flat pencil of metrics and the theory of Frobenius manifolds outlined in [14].

Let M be a smooth manifold of dimension r . A symmetric bilinear form (\cdot, \cdot) on T^*M is called **contravariant metric** if it is invertible on an open dense subset $M_0 \subset M$. In a local coordinates (u^1, \dots, u^r) , if we set

$$g^{ij}(u) = (du^i, du^j); \quad i, j = 1, \dots, r, \quad (2.1)$$

Then the inverse matrix $g_{ij}(u)$ of $g^{ij}(u)$ determines a metric $\langle \cdot, \cdot \rangle$ on TM_0 . We define the **contravariant Levi-Civita connection** Γ_k^{ij} for (\cdot, \cdot) by

$$\Gamma_k^{ij} := -g^{is}\Gamma_{sk}^j \quad (2.2)$$

where Γ_{sk}^j is the Levi-Civita connection of $\langle \cdot, \cdot \rangle$. We say the metric (\cdot, \cdot) is flat if $\langle \cdot, \cdot \rangle$ is flat.

Let $g_1^{ij}(u)$ and $g_2^{ij}(u)$ be two contravariant flat metrics on M and denote the corresponding Levi-Civita connections by $\Gamma_{1;k}^{ij}(u)$ and $\Gamma_{2;k}^{ij}(u)$, respectively. We say $g_1^{ij}(u)$ and $g_2^{ij}(u)$ form a **flat pencil of metrics** if

- (1) $g_\lambda^{ij}(u) := g_2^{ij}(u) + \lambda g_1^{ij}(u)$ defines a flat metric on T^*M for a generic λ and,
- (2) The Levi-Civita connection of $g_\lambda^{ij}(u)$ is given by

$$\Gamma_{\lambda;k}^{ij}(u) = \Gamma_{2;k}^{ij}(u) + \lambda \Gamma_{1;k}^{ij}(u).$$

The flat pencil of metrics in this work is obtained by using the following lemma

Lemma 2.1. [12] *If for a contravariant flat metric g_2^{ij} in some coordinate (u^1, \dots, u^r) the entries of $g_2^{ij}(u)$ and its Levi-Civita connection $\Gamma_{2;k}^{ij}$ depend linearly on u^r then the metric*

$$g_1^{ij} = \partial_{u^r} g_2^{ij} \quad (2.3)$$

with g_2^{ij} form a flat pencil of metrics. The Levi-Civita connection of the metric g_1^{ij} has the form

$$\Gamma_{1;k}^{ij} = \partial_{u^r} \Gamma_{2;k}^{ij}. \quad (2.4)$$

We are concern with the following particular class of flat pencil of metrics.

Definition 2.2. A contravariant flat pencil of metrics on a manifold M defined by the matrices g_1^{ij} and g_2^{ij} is called **quasihomogenous of degree d** if there exists a function τ on M such that the vector fields

$$\begin{aligned} E &:= \nabla_2 \tau, \quad E^i = g_2^{is} \partial_s \tau \\ e &:= \nabla_1 \tau, \quad e^i = g_1^{is} \partial_s \tau \end{aligned} \quad (2.5)$$

satisfy the following properties

- (1) $[e, E] = e$.
- (2) $\mathfrak{L}_E(\cdot, \cdot)_2 = (d-1)(\cdot, \cdot)_2$.
- (3) $\mathfrak{L}_e(\cdot, \cdot)_2 = (\cdot, \cdot)_1$.
- (4) $\mathfrak{L}_e(\cdot, \cdot)_1 = 0$.

Here, for example \mathfrak{L}_E denote the Lie derivative along the vector field E and $(\cdot, \cdot)_1$ denote the metric defined by the matrix g_1^{ij} . In addition, the quasihomogenous flat pencil of metrics is called **regular** if the (1,1)-tensor

$$R_i^j = \frac{d-1}{2} \delta_i^j + \nabla_{1i} E^j \quad (2.6)$$

is nondegenerate on M .

2.1.1. Frobenius manifolds. A Frobenius algebra is a commutative associative algebra with unity e and an invariant nondegenerate bilinear form (\cdot, \cdot) . A **Frobenius manifold** is a manifold M with a smooth structure of Frobenius algebra on the tangent space $T_t M$ at any point $t \in M$ with certain compatibility conditions [13]. Globally, we require the metric (\cdot, \cdot) to be flat and the unity vector field e is constant with respect to it. In the flat coordinates (t^1, \dots, t^r) where $e = \frac{\partial}{\partial t^{r-1}}$ the compatibility conditions implies that there exist a function $\mathbb{F}(t^1, \dots, t^r)$ such that

$$\eta_{ij} = (\partial_{t^i}, \partial_{t^j}) = \partial_{t^{r-1}} \partial_{t^i} \partial_{t^j} \mathbb{F}(t)$$

and the structure constants of the Frobenius algebra is given by

$$C_{ij}^k = \eta^{kp} \partial_{t^p} \partial_{t^i} \partial_{t^j} \mathbb{F}(t)$$

where η^{ij} denote the inverse of the matrix η_{ij} . In this work, we consider Frobenius manifolds where the quasihomogeneity condition takes the form

$$\sum_{i=1}^r d_i t^i \partial_{t^i} \mathbb{F}(t) = (3-d) \mathbb{F}(t); \quad d_{r-1} = 1. \quad (2.7)$$

This condition defines **the degrees d_i** and **the charge d** of the Frobenius structure. If $\mathbb{F}(t)$ is an algebraic function we call M an **algebraic Frobenius manifold**. The associativity of Frobenius algebra implies the potential $\mathbb{F}(t)$ satisfy a system of partial differential equations which appears in topological field theory and called WDVV equations:

$$\partial_{t^i} \partial_{t^j} \partial_{t^k} \mathbb{F}(t) \eta^{kp} \partial_{t^p} \partial_{t^q} \partial_{t^n} \mathbb{F}(t) = \partial_{t^n} \partial_{t^j} \partial_{t^k} \mathbb{F}(t) \eta^{kp} \partial_{t^p} \partial_{t^q} \partial_{t^i} \mathbb{F}(t). \quad (2.8)$$

The following theorem gives a connection between the geometry of Frobenius manifolds and flat pencil of metrics.

Theorem 2.3. [14] *A contravariant quasihomogenous regular flat pencil of metrics of degree d on a manifold M defines a Frobenius structure on M of degree d .*

It is well known that from a Frobenius manifold we always have a flat pencil of metrics but it does not necessary satisfy the regularity condition (2.6) [14]. Locally, in the coordinates defining equation (2.8), the flat pencil of metrics is found from the equations

$$\begin{aligned}\eta^{ij} &= g_1^{ij} \\ g_2^{ij} &= (d - 1 + d_i + d_j)\eta^{i\alpha}\eta^{j\beta}\partial_\alpha\partial_\beta\mathbb{F}\end{aligned}\tag{2.9}$$

This flat pencil of metric is quasihomogenous of degree d with $\tau = t^1$. Furthermore, we have

$$E = \sum_i d_i t^i \partial_{t^i}; \quad e = \partial_{t^{r-1}}.\tag{2.10}$$

2.2. Local Poisson brackets. In this section we fix notations and we review the Dirac reduction for local Poisson brackets on loop spaces.

Let M be a manifold. The loop spaces $\mathfrak{L}(M)$ of M is the space of smooth functions from the circle to M . A local Poisson bracket $\{.,.\}$ is a Poisson bracket on the space of local functional on $\mathfrak{L}(M)$. If we choose a local coordinates (u^1, \dots, u^r) then $\{.,.\}$ is a finite summation of the form

$$\begin{aligned}\{u^i(x), u^j(y)\} &= \sum_{k=-1}^{\infty} \epsilon^k \{u^i(x), u^j(y)\}^{[k]} \\ \{u^i(x), u^j(y)\}^{[k]} &= \sum_{s=0}^{k+1} A_{k,s}^{i,j}(u(x)) \delta^{(k-s+1)}(x-y),\end{aligned}\tag{2.11}$$

where ϵ is just a parameter, $A_{k,s}^{i,j}(u(x))$ are homogenous polynomials in $\partial_x^j u^i(x)$ of degree s when we assign $\partial_x^j u^i(x)$ degree j , and $\delta(x-y)$ is the Dirac delta function defined by

$$\int_{S^1} f(y) \delta(x-y) dy = f(x).$$

In particular, the first terms can be written as follows

$$\begin{aligned}\{u^i(x), u^j(y)\}^{[-1]} &= F^{ij}(u(x)) \delta(x-y) \\ \{u^i(x), u^j(y)\}^{[0]} &= g^{ij}(u(x)) \delta'(x-y) + \Gamma_k^{ij}(u(x)) u_x^k \delta(x-y).\end{aligned}\tag{2.12}$$

where $g^{ij}(u)$, $F^{ij}(u)$ and $\Gamma_k^{ij}(u)$ are smooth functions on the finite dimensional space M . It follows from the definition that the matrix $F^{ij}(u)$ defines a Poisson structure on M . We say the Poisson bracket admits a **dispersionless limit** if $F^{ij}(u) = 0$ and $\{u^i(x), u^j(y)\}^{[0]} \neq 0$. In this case $\{u^i(x), u^j(y)\}^{[0]}$ defines a Poisson bracket on $\mathfrak{L}(M)$ known as **Poisson bracket of hydrodynamic type**. We call it **nondegenerate** if $\det(g^{ij}(u)) \neq 0$ in an open dense subset of M .

Theorem 2.4. [15] *In the notations given above. If $\{u^i(x), u^j(y)\}^{[0]}$ defines a nondegenerate Poisson brackets of hydrodynamic type then the matrix $g^{ij}(u)$ defines a contravariant flat metric on M and $\Gamma_k^{ij}(u)$ is its contravariant Levi-Civita connection.*

2.2.1. Dirac reduction. Assume we have a local Poisson bracket on the loop space $\mathfrak{L}(M)$ of a manifold M . Let $N \subset M$ be a submanifold of dimension m . Then under some assumptions the Poisson bracket can be reduced to N using Dirac reduction. For this end we assume N is defined by the equations $u^\alpha = 0$ for $\alpha = m+1, \dots, r$. We introduce three types of indexes; capital letters $I, J, K, \dots = 1, \dots, r$, small letters $i, j, k, \dots = 1, \dots, m$ which parameterize the submanifold N and Greek letters $\alpha, \beta, \delta, \dots = m+1, \dots, r$.

Proposition 2.5. [8] *In the notations of equations (2.12). Assume the minor matrix $F^{\alpha\beta}$ is nondegenerate. Then Dirac reduction is well defined on $\mathfrak{L}(N)$ and gives a local Poisson bracket. If we write the leading terms of the reduced Poisson bracket in the form*

$$\{u^i(x), u^j(y)\}_N^{[-1]} = \tilde{F}^{ij}(u)\delta(x-y), \quad (2.13)$$

$$\{u^i(x), u^j(y)\}_N^{[0]} = \tilde{g}_0^{ij}(u)\delta'(x-y) + \tilde{\Gamma}_k^{ij} u_x^k \delta(x-y). \quad (2.14)$$

Then

$$\tilde{F}^{ij} = (F^{ij} - F^{i\beta} F_{\beta\alpha} F^{\alpha j}), \quad (2.15)$$

$$\tilde{g}^{ij} = g_0^{ij} - g^{i\beta} F_{\beta\alpha} F^{\alpha j} + F^{i\beta} F_{\beta\alpha} g^{\alpha\varphi} F_{\varphi\gamma} F^{\gamma j} - F^{i\beta} F_{\beta\alpha} g^{\alpha j}, \quad (2.16)$$

and

$$\begin{aligned} \tilde{\Gamma}_k^{ij} u_x^k &= (\Gamma_k^{ij} - \Gamma_k^{i\beta} F_{\beta\alpha} F^{\alpha j} + F^{i\lambda} F_{\lambda\alpha} \Gamma_k^{\alpha\beta} F_{\beta\varphi} F^{\varphi j} - F^{i\beta} F_{\beta\alpha} \Gamma_k^{\alpha j}) u_x^k \\ &\quad - (g^{i\beta} - F^{i\lambda} F_{\lambda\alpha} g^{\alpha\beta}) \partial_x (F_{\beta\varphi} F^{\varphi j}) \end{aligned} \quad (2.17)$$

and the other terms could be found by solving certain recursive equations.

Corollary 2.6. *If the entries $F^{i\alpha} = 0$ on N , then the reduced Poisson bracket on $\mathfrak{L}(N)$ will have the same leading terms, i.e*

$$\begin{aligned} \tilde{F}^{ij} &= F^{ij}. \\ \tilde{g}^{ij} &= g^{ij}. \\ \tilde{\Gamma}_k^{ij} &= \Gamma_k^{ij}. \end{aligned} \quad (2.18)$$

3. SUBREGULAR NILPOTENT ELEMENTS

We review some facts about the theory of subregular nilpotent elements in simple Lie algebras and the related structure of opposite Cartan subalgebra.

Let \mathfrak{g} be simple Lie algebra of rank r . We assume the Lie algebra \mathfrak{g} is of type D_r where r is even or E_r . This assumption is due to the fact that for simply laced Lie algebra, the opposite Cartan subalgebra for a subregular nilpotent element exists only for these types of Lie algebra.

Let us fix a subregular nilpotent element $e \in \mathfrak{g}$. By definition, a nilpotent element is called subregular if $\mathfrak{g}^e := \ker \text{ad } e$ has dimension equal to $r + 2$ [4]. Using the Jacobson-Morozov theorem, we fix a semisimple element h and a nilpotent element f in \mathfrak{g} such that $\{e, h, f\}$ generate a sl_2 -subalgebra $\mathcal{A} \subset \mathfrak{g}$ satisfying

$$[h, e] = 2e, [h, f] = -2f, [e, f] = h. \quad (3.1)$$

Let us consider the adjoint representation of \mathcal{A} on \mathfrak{g} . Then \mathfrak{g} decomposes to irreducible \mathcal{A} -submodules

$$\mathfrak{g} = \bigoplus_{i=1}^{r+2} V^i. \quad (3.2)$$

Let $\dim V^i = 2\eta_i + 1$ and assume V^1 is isomorphic to \mathcal{A} as a vector space. We call the set

$$Wt(e) = \{\eta_i : i = 1, \dots, r+2\} \quad (3.3)$$

the weights of e . Let $\eta_0 + 1$ be the Coxeter number of \mathfrak{g} . The set $Wt(e)$, under our choice of a total order, together with η_0 are given in the following table.

\mathfrak{g}	$Wt(e)$			
	η_0	$\eta_1, \dots, \eta_{r-1}$	η_r	η_{r+1}, η_{r+2}
D_r	$2r-3$	$1, 3, \dots, r-1; r-1, r+1, \dots, 2r-5$	r-3	$1, r-2$
E_6	11	$1, 4, 5, 7, 8$	2	$3, 5$
E_7	17	$1, 5, 7, 9, 11, 13$	3	$5, 8$
E_8	29	$1, 7, 11, 13, 17, 19, 23$	5	$9, 14$
	$Et(\mathfrak{g})$			

We observe that the set $Et(\mathfrak{g})$ of exponents of \mathfrak{g} is given by

$$Et(\mathfrak{g}) = \{\eta_i : i = 0, \dots, r-1\}. \quad (3.4)$$

We emphasize that the statements and proofs in this work depend explicitly on the total ordering of the set $Wt(e)$ and $Et(\mathfrak{g})$ given in this table.

We fix on \mathfrak{g} the invariant bilinear form $\langle \cdot | \cdot \rangle$ such that $\langle e | f \rangle = 1$. We normalize the decomposition (3.2) and we fix a basis for each V^i by using the following proposition.

Proposition 3.1. *There exists a decomposition of \mathfrak{g} into a sum of irreducible \mathcal{A} -submodules*

$$\mathfrak{g} = \bigoplus_{i=1}^{r+2} V^i$$

in such a way that there is a basis $X_I^i, I = -\eta_i, -\eta_i + 1, \dots, \eta_i$ for each $V^i, i = 1, \dots, r+2$ satisfying the following relations

$$X_I^i = \frac{1}{(\eta_i + I)!} \text{ad} e^{\eta_i + I} X_{-\eta_i}^i, \quad I = -\eta_i, -\eta_i + 1, \dots, \eta_i. \quad (3.5)$$

and

$$\begin{aligned} \text{ad} h X_I^i &= 2IX_I^i. \\ \text{ad} e X_I^i &= (\eta_i + I + 1)X_{I+1}^i. \\ \text{ad} f X_I^i &= (\eta_i - I + 1)X_{I-1}^i. \end{aligned} \quad (3.6)$$

Furthermore

$$\langle X_I^i, X_J^j \rangle = \delta_{i,j} \delta_{I,-J} (-1)^{\eta_i - I + 1} \binom{2\eta_i}{\eta_i - I}. \quad (3.7)$$

Proof. The proof is similar to the proof of proposition 2.3 in [10], since for all simple Lie algebras, except D_4 , there are at most two weights of the same value. The Lie algebra D_4 has three weights equal one. We give such a normalization for D_4 in section 6.1. \square

We observe that the normalized basis for V^1 are

$$X_1^1 = -e, \quad X_0^1 = h, \quad X_{-1}^1 = f.$$

We recall that the semisimple element h define the following \mathbb{Z} -grading on \mathfrak{g} and it is called **the Dynkin grading**

$$\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i; \quad \mathfrak{g}_i = \{q \in \mathfrak{g} : \text{ad} h(q) = iq\}. \quad (3.8)$$

We observe that $\mathfrak{g}_i = 0$ if i is odd.

3.1. Opposite Cartan subalgebra. The following theorem summarize the relation between the subregular nilpotent element e and a quasi-Coxeter conjugacy class in the Weyl group of \mathfrak{g} . To simplify the notations let κ **denote the maximum weight** η_{r-1} .

Theorem 3.2. *There exists a nonzero element $X' \in \mathfrak{g}_{-2\kappa}$ such that the element*

$$y_1 = e + X'$$

is regular semisimple. Let \mathfrak{h}' be the Cartan subalgebra containing y_1 , i.e

$$\mathfrak{h}' = \ker \operatorname{ad} y_1$$

and consider the adjoint group element w defined by

$$w = \exp \frac{2\pi \mathbf{i}}{\kappa + 1} \operatorname{ad} h.$$

Then w acts on \mathfrak{h}' as a representative of a regular quasi-Coxeter conjugacy class in the Weyl group acting on \mathfrak{h}' . The conjugacy class is of type $D_r(a_1)$ (resp. $E_r(a_1)$) if \mathfrak{g} is of type D_r (resp. E_r). Furthermore, the element y_1 can be completed to a basis y_i , $i = 1, \dots, r$ for \mathfrak{h}' having the form

$$y_i = v_i + u_i, \quad u_i \in \mathfrak{g}_{2\eta_i}, \quad v_i \in \mathfrak{g}_{2\eta_i - 2(\kappa+1)}$$

and such that y_i is an eigenvector of w with eigenvalue $\exp \frac{2\pi \mathbf{i} \eta_i}{\kappa+1}$.

Proof. The proof for a Lie algebra of type E_i , $i = 6, 7, 8$ is obtained by Springer [27]. The case of a Lie algebra of type D_r , the proof is given in the appendix of [7]. \square

We fix an element X' satisfy the hypothesis of the theorem above. In the literature, the element $y_1 = e + X'$ is called a **cyclic element** and $\mathfrak{h}' = \ker \operatorname{ad} y_1$ is called the **opposite Cartan subalgebra**. We will call the set

$$Et(e) := \{\eta_i, i = 1, \dots, r\} \subset Wt(e) \quad (3.9)$$

the exponents of e as it plays the same role of the exponents of (\mathfrak{g}) to the regular nilpotent elements [10]. Let us consider a basis $y_i = u_i + v_i$ for \mathfrak{h}' satisfying the hypothesis of the theorem above. We normalize them by using the following theorem

Proposition 3.3. *The basis $y_i = u_i + v_i$ can be chosen in such a way that*

$$u_i = -X_{\eta_i}^i, \quad i = 1, \dots, r \quad (3.10)$$

and

$$v_1 = X' = \rho X_{-\kappa}^{r-1} \quad (3.11)$$

for some nonzero number ρ

Proof. From the construction we know that $u_1 = e = -X_1^1$. Then, it is easy to see that u_i , $i = 1, \dots, r$ generate a commutative subalgebra of \mathfrak{g}^e . But $X_{\eta_i}^i$ are homogenous basis for \mathfrak{g}^e . Hence for a Lie algebra of type E_8 and D_r , $r > 4$ the normalization of u_i , $i = 2, \dots, r$ and v_1 follows from the structure of the set $Wt(e)$. For a Lie algebra of type E_6 , E_7 and D_4 we obtained such normalization by direct computations. \square

Let us consider the matrix $A_{i,j}$ of the invariant bilinear form on \mathfrak{h}' under the basis $y_i = -X_{\eta_i}^i + v_i$.

$$A_{ij} = \langle y_i | y_j \rangle = -\langle X_{\eta_i}^i | v_j \rangle - \langle v_i | X_{\eta_j}^j \rangle; \quad i, j = 1, \dots, r. \quad (3.12)$$

We know from the theory of Cartan subalgebras that the matrix A_{ij} is nondegenerate. Some useful properties we gain from \mathfrak{h}' are summarized in the following proposition.

Proposition 3.4. *The matrix A_{ij} is antidiagonal with respect to the set $Et(e)$ in the sense that*

$$A_{ij} = 0, \text{ if } \eta_i + \eta_j \neq \kappa + 1.$$

Therefore, after totally reordering the set $Et(e)$ in the form

$$\mu_1 \leq \mu_2 \leq \dots \leq \mu_r,$$

we have the property

$$\mu_j + \mu_{r-j+1} = \kappa + 1 = \eta_1 + \eta_{r-1}, j = 1, \dots, r. \quad (3.13)$$

Proof. We will use the fact that the matrix $\langle \cdot | \cdot \rangle$ is a nondegenerate invariant bilinear form on \mathfrak{h}' . Hence for any element y_i there exists an element y_j such that $\langle y_i | y_j \rangle \neq 0$. But if w is the quasi-Coxeter element we defined in theorem 3.2 then the equality

$$\langle y_i | y_j \rangle = \langle wy_i | wy_j \rangle = \exp \frac{2(\eta_i + \eta_j)\pi \mathbf{i}}{\kappa + 1} \langle y_i | y_j \rangle$$

implies that in case $\langle y_i | y_j \rangle \neq 0$ we must have $\eta_i + \eta_j = \kappa + 1$. Hence, the matrix A_{ij} is antidiagonal with respect to the set $Et(e)$. \square

In the remainder of this paper let a **denote the element** $X_{-\kappa}^{r-1}$.

Proposition 3.5. *The commutators of a and $X_{\eta_i}^i$ satisfy the relation*

$$\frac{\langle [a, X_{\eta_i}^i] | X_{\eta_j-1}^j \rangle}{2\eta_j} + \frac{\langle [a, X_{\eta_j}^j] | X_{\eta_i-1}^i \rangle}{2\eta_i} = \frac{1}{\rho} A_{ij} \quad (3.14)$$

for all $i, j = 1, \dots, r$. Here the nonzero number ρ is the same as in proposition 3.3.

Proof. We note that the commutator of $y_1 = e + \rho X_{-\kappa}^{r-1}$ and $y_i = v_i - X_{\eta_i}^i$ gives the relation

$$[e, v_i] = \rho [a, X_{\eta_i}^i], \quad i = 1, \dots, r. \quad (3.15)$$

This in turn give the following equality for every $i, j = 1, \dots, r$

$$\begin{aligned} \langle \rho [a, X_{\eta_i}^i] | X_{\eta_j-1}^j \rangle &= \langle [e, v_i] | X_{\eta_j-1}^j \rangle = -\langle v_i | [e, X_{\eta_j-1}^j] \rangle \\ &= -2\eta_j \langle v_i | X_{\eta_j}^j \rangle \end{aligned} \quad (3.16)$$

but then

$$\frac{\langle [a, X_{\eta_i}^i] | X_{\eta_j-1}^j \rangle}{2\eta_j} + \frac{\langle [a, X_{\eta_j}^j] | X_{\eta_i-1}^i \rangle}{2\eta_i} = -\frac{1}{\rho} (\langle v_i | X_{\eta_j}^j \rangle + \langle v_j | X_{\eta_i}^i \rangle) = \frac{1}{\rho} A_{ij}. \quad (3.17)$$

\square

3.2. Slodowy coordinates. We review the relation between subregular nilpotent elements and deformation of simple hypersurface singularities [26].

Let G be the adjoint group of \mathfrak{g} . By Chevalley theorem, the algebra $S(\mathfrak{g}^*)^G$ of invariant polynomials under the adjoint action of G is generated by r homogenous polynomials of degrees $\eta_i + 1$, $i = 0, \dots, r-1$. The inclusion homomorphism

$$S(\mathfrak{g}^*)^G \hookrightarrow S(\mathfrak{g}^*),$$

is dual to a morphism

$$\chi : \mathfrak{g} \rightarrow \mathfrak{g}/G$$

called **the adjoint quotient**. We fix a homogenous generator $\chi^0, \dots, \chi^{r-1}$ of the algebra $S(\mathfrak{g}^*)^G$ with degree χ^i equals $\eta_i + 1$. Then the adjoint quotient map is given by

$$\chi = (\chi^0, \dots, \chi^{r-1}) : \mathfrak{g} \rightarrow \mathbb{C}^r. \quad (3.18)$$

The fiber $\mathcal{N} := \chi^{-1}(\chi(0))$ is called the **nilpotent variety of \mathfrak{g}** , it consists of all nilpotent elements of \mathfrak{g} .

We define the **Slodowy slice** to be the affine subspace

$$Q = e + \mathfrak{g}^f \quad (3.19)$$

where $\mathfrak{g}^f = \ker \operatorname{ad} f$. Then Brieskorn proved that the restriction of χ to Q is semiuniversal deformation of the simple hypersurface singularity $\mathcal{N} \cap Q$ which is of the same type as \mathfrak{g} . Let us introduce the following coordinates on Q

$$\sum_{i=1}^{r+2} z^i X_{-\eta_i}^i + e \in Q$$

and assign degree $2\eta_i + 2$ to z^i .

Proposition 3.6. ([25], section 7.2) *Let χ^i , $i = 0, \dots, r-1$ be a homogenous generators of the ring $S(\mathfrak{g}^*)^G$. Then the restriction of χ^i to Q will be quasihomogenous of degree $2\eta_i + 2$.*

Proposition 3.7. *The map $\chi|_Q$ has rank $r-1$ at e . We can normalize the modules V^i and choose a homogenous generators $\chi^0, \dots, \chi^{r-1}$ for $S(\mathfrak{g}^*)^G$ such that the restriction t^i of χ^i to Q with $i > 0$ take the form*

$$t^i = z^i + \text{non linear terms.} \quad (3.20)$$

In particular, setting

$$t^{r+i} = z^{r+i}, \quad i = 0, 1, 2 \quad (3.21)$$

we get a quasihomogenous coordinates (t^1, \dots, t^{r+2}) on Q with degree t^i equals degree z^i .

Proof. Let $\chi^0, \dots, \chi^{r-1}$ be a homogenous generators for $S(\mathfrak{g}^*)^G$ and denote t^1, \dots, t^{r-1} their restriction to Q . Using the structure of the set $Wt(e)$ and $Et(\mathfrak{g})$ together with the isolatedness of the singularity of $\mathcal{N} \cap Q$, Slodowy proved the following [25] (section 8.3). We can choose a coordinates v^1, v^2, v^3 from z^1, \dots, z^{r+2} of degrees $2\eta_r + 2, 2\eta_{r+1} + 2, 2\eta_{r+2} + 2$, respectively, such that

$$(t^1, \dots, t^{r-1}, v^1, v^2, v^3)$$

are homogenous coordinates on Q . Therefore, the statement follows upon proving that $v_i = z^{r+i}$ for $i = 0, 1, 2$ when considering the normalization of proposition 3.1. This is obvious for the Lie algebra E_8 since the numbers in the set $Wt(e)$ are all different. It is also true for the Lie algebra D_r , r is even, since the restriction of the first invariant χ^1 to S is, up to constant, equal to z^1 . For the Lie algebra E_6 (respectively E_7) we verify by direct computation that the invariant χ^3 (resp. χ^2) depends explicitly on z^3 (resp. z^2). \square

We will call the coordinates (z^1, \dots, z^{r+2}) on Q obtained in this proposition **Slodowy coordinates**. We observe that in this coordinates the quotient map take the form

$$\chi|_Q : (t^1, \dots, t^{r+2}) \mapsto (t^0, t^1, \dots, t^{r-1}) \quad (3.22)$$

where t^0 is the restriction to Q of the invariant polynomial χ^0 . Setting t^1, \dots, t^r equal zero in t^0 we get, from the quasihomogeneity, a polynomial function $f(t^r, t^{r+1}, t^{r+2})$ of the form shown in the table below (we lower the index for convenience). Here c_1, c_2, c_3, c_4 are some constants. Note that the hypersurface $\mathcal{N} \cap Q$ will be given by setting $f(t^r, t^{r+1}, t^{r+2}) = 0$. Moreover, from the isolatedness of the singularity the numbers c_1, c_2, c_3 are nonzero constants. The constant c_4 could be eliminated by change of variables to obtain the standard equation defining the simple hypersurface singularity.

\mathfrak{g}	$f(t_r, t_{r+1}, t_{r+2})$
D_r	$c_1 t_{r+1}^{r-1} + c_2 t_{r+1} t_r^2 + c_3 t_{r+2}^2 + c_4 t_{r+1}^{\frac{r}{2}} t_r$
E_6	$c_1 t_6^4 + c_2 t_7^3 + c_3 t_8^2 + c_4 t_6^2 t_8$
E_7	$c_1 t_7^3 t_8 + c_2 t_8^3 + c_3 t_9^2$
E_8	$c_1 t_8^5 + c_2 t_9^3 + c_3 t_{10}^2$

4. DRINFELD-SOKOLOV REDUCTION

In this section we review the construction of the classical W -algebra associated to the nilpotent element e using Drinfeld-Sokolov reduction.

We use the Dynkin grading of e to define the following subalgebras

$$\begin{aligned} \mathfrak{b} &:= \bigoplus_{i \leq 0} \mathfrak{g}_i, \\ \mathfrak{n} &:= \bigoplus_{i \leq -2} \mathfrak{g}_i = [\mathfrak{b}, \mathfrak{b}]. \end{aligned} \quad (4.1)$$

Then we consider the action of the adjoint group \mathcal{N} of $\mathfrak{L}(\mathfrak{n})$ on $\mathfrak{L}(\mathfrak{g})$ defined by

$$q(x) \rightarrow \exp \operatorname{ad} s(x)(\partial_x + q(x)) - \partial_x \quad (4.2)$$

where $s(x) \in \mathfrak{L}(\mathfrak{n})$, $q(x) \in \mathfrak{L}(\mathfrak{g})$.

Let us extend the invariant bilinear form from \mathfrak{g} to $\mathfrak{L}(\mathfrak{g})$ by setting

$$(u|v) = \int_{S^1} \langle u(x)|v(x) \rangle dx, \quad u, v \in \mathfrak{L}(M). \quad (4.3)$$

Then we identify $\mathfrak{L}(\mathfrak{g})$ with $\mathfrak{L}(\mathfrak{g})^*$ by means of this bilinear form. We define the gradient $\delta \mathcal{F}(q)$ for a functional \mathcal{F} on $\mathfrak{L}(\mathfrak{g})$ to be the unique element in $\mathfrak{L}(\mathfrak{g})$ satisfying

$$\frac{d}{d\theta} \mathcal{F}(q + \theta \dot{s})|_{\theta=0} = \int_{S^1} \langle \delta \mathcal{F} | \dot{s} \rangle dx \text{ for all } \dot{s} \in \mathfrak{L}(\mathfrak{g}). \quad (4.4)$$

We fix on $\mathfrak{L}(\mathfrak{g})$ the following Poisson bracket

$$\{\mathcal{F}[q(x)], \mathcal{I}[q(y)]\} = \frac{1}{\epsilon} (\delta \mathcal{F}(x) | [\epsilon \partial_x + q(x), \delta \mathcal{I}(x)]) \quad (4.5)$$

for every functional \mathcal{F} and \mathcal{I} on $\mathfrak{L}(\mathfrak{g})$.

Proposition 4.1. ([8]) *The action of \mathcal{N} on $\mathfrak{L}(\mathfrak{g})$ with Poisson bracket $\{.,.\}$ is Hamiltonian. It admits a momentum map J to be the projection*

$$J : \mathfrak{L}(\mathfrak{g}) \rightarrow \mathfrak{L}(\mathfrak{n}^+)$$

where \mathfrak{n}^+ is the image of \mathfrak{n} under the Killing map. Moreover, J is Ad^* -equivariant.

We take e as a regular value of J . Since \mathfrak{b} is the orthogonal complement to \mathfrak{n} under $\langle .|. \rangle$, we get the affine space $S = J^{-1}(e) = \mathfrak{L}(\mathfrak{b}) + e$. Moreover, it follows from the Dynkin grading that the isotropy group of e is \mathcal{N} . Let R be the ring of invariant differential polynomials of S under the action of \mathcal{N} . Then, from Marsden-Ratiu reduction theorem, the set \mathcal{R} of functionals on S which have densities in the ring R is closed under the Poisson brackets $\{.,.\}$.

Let us define the space \tilde{Q} to be the Slodowy slice

$$\tilde{Q} := e + \mathfrak{L}(\mathfrak{g}^f). \quad (4.6)$$

The following proposition identifies the space S/\mathcal{N} with \tilde{Q} .

Proposition 4.2. [8] *The space \tilde{Q} is a cross section for the action of \mathcal{N} on S , i.e for any element $q(x) + e \in S$ there is a unique element $s(x) \in \mathfrak{L}(\mathfrak{n})$ such that*

$$z(x) + e = \exp \operatorname{ad} s(x)(\partial_x + q(x)) - \partial_x \in \tilde{Q}. \quad (4.7)$$

Therefore, the entries of $z(x)$ are generators of the ring R .

Hence, the space \tilde{Q} has a Poisson structure $\{.,.\}^{\tilde{Q}}$ from $\{.,.\}$. This Poisson bracket is known in the literature as **classical W -algebra** associated to e . For a formal definition of classical W -algebras see [20].

Let us obtain the linear terms of the invariants $z^i(x)$. We introduce a parameter τ and write

$$\begin{aligned} q(x) + e &= \tau \sum_{i=1}^{r+2} \sum_{I=0}^{\eta_i} q_i^I X_{-I}^i + e \in S \\ z(x) + e &= \tau \sum_{i=1}^{r+2} z^i(x) X_{-\eta_i}^i + e \in \tilde{Q} \\ s(x) &= \tau \sum_{i=1}^{r+2} \sum_{I=1}^{\eta_i} s_i^I(x) X_{-I}^i \in \mathfrak{L}(\mathfrak{n}). \end{aligned}$$

Then equation (4.7) expands to

$$\begin{aligned} \sum_{i=1}^{r+2} z^i(x) X_{-\eta_i}^i + \sum_{i=1}^{r+2} \sum_{I=1}^{\eta_i} (\eta_i - I + 1) s_i^I X_{-I+1}^i = \\ \sum_{i=1}^{r+2} \sum_{I=0}^{\eta_i} q_i^I(x) X_{-I}^i - \sum_{i=1}^{r+2} \sum_{I=1}^{\eta_i} \partial_x s_i^I(x) X_{-I}^i + \mathcal{O}(\tau). \end{aligned} \quad (4.8)$$

Hence, any invariant $z^i(x)$ will take the form

$$\begin{aligned} z^i(x) &= q_i^{\eta_i} - \partial_x s_i^{\eta_i} + \mathcal{O}(\tau) \\ &= q_i^{\eta_i}(x) - \partial_x q_i^{\eta_i-1} + \mathcal{O}(\tau). \end{aligned} \quad (4.9)$$

Furthermore, using $\langle e|f \rangle = 1$ we get

$$\begin{aligned} z^1(x) &= q_1^1(x) - \partial_x s_1^1 + \tau \langle e|[s_1^1(x) X_{-1}^1, q_1^0 X_0^1] \rangle \\ &\quad + \frac{1}{2} \tau \langle e|[s_1^1(x) X_{-1}^1, [s_1^1(x) X_{-1}^1, e]] \rangle \\ &= q_1^1(x) - \partial_x q_1^0(x) + \frac{1}{2} \tau \langle e|[s_1^1(x) X_{-1}^1, q_1^0 X_0^1] \rangle \\ &= q_1^1(x) - \partial_x q_1^0(x) + \frac{1}{2} \tau \sum_i (q_i^0(x))^2 \langle X_0^i | X_0^i \rangle. \end{aligned} \quad (4.10)$$

The invariant $z^1(x)$ is known in the literature as the **Virasoro density**.

We observe that the reduced Poisson structure could be obtained as follows. We write the coordinates of \tilde{Q} as a differential polynomials in the coordinates of S using equation (4.7) and then we apply the Leibnitz rule. The Leibnitz rule for $u, v \in R$ have the following form

$$\{u(x), v(y)\} = \frac{\partial u(x)}{\partial (q_i^I)^{(m)}} \partial_x^m \left(\frac{\partial v(y)}{\partial (q_j^J)^{(n)}} \partial_y^n (\{q_i^I(x), q_j^J(y)\}) \right). \quad (4.11)$$

Our analysis for the Poisson brackets will rely on the quasihomogeneity of the invariants $z^i(x)$ in the coordinates of $q(x) \in \mathfrak{L}(\mathfrak{b})$ and their derivatives.

Lemma 4.3. *If we assign degree $2J+2l+2$ to $\partial_x^l(q_i^J(x))$ then $z^i(x)$ will be quasihomogenous of degree $2\eta_i + 2$. Furthermore, each invariant $z^i(x)$ depends linearly only on $q_i^{\eta_i}(x)$ and $\partial_x q_i^{\eta_i-1}(x)$, i.e*

$$z^i(x) = q_i^{\eta_i}(x) - \partial_x q_i^{\eta_i-1} + \text{nonlinear terms.} \quad (4.12)$$

Furthermore

$$z^1(x) = q_1^1(x) - \partial_x q_1^0(x) + \frac{1}{2} \sum_i (q_i^0(x))^2 \langle X_0^i | X_0^i \rangle. \quad (4.13)$$

In particular, $z^i(x)$ with $i \neq r-1$ does not depend on $q_{r-1}^\kappa(x)$ or its derivatives.

We fix the following notations for the leading terms of the Poisson bracket

$$\{z^i(x), z^j(y)\}^{\tilde{Q}} = \sum_{k=-1}^{\infty} \epsilon^k \{z^i(x), z^j(y)\}_1^{[k]} \quad (4.14)$$

where

$$\begin{aligned} \{z^i(x), z^j(y)\}^{[-1]} &= F^{ij}(z(x))\delta(x-y) \\ \{z^i(x), z^j(y)\}^{[0]} &= g^{ij}(z(x))\delta'(x-y) + \Gamma_k^{ij}(z(x))z_x^k\delta(x-y) \end{aligned} \quad (4.15)$$

4.1. The nondegeneracy condition. We want to prove that the minor matrix $g^{mn}(z)$, $m, n = 1, \dots, r$ is nondegenerate generically on \tilde{Q} . For this end we define the matrix

$$g_1^{ij}(z) = \partial_{z^{r-1}} g^{ij}(z) \quad (4.16)$$

and we will prove that the minor matrix $g_1^{mn}(z)$, $m, n = 1, \dots, r$ is lower antidiagonal with respect to the set $Et(e)$, i.e

$$g_1^{mn} = \begin{cases} 0 & \text{if } \eta_m + \eta_n < \kappa + 1 \\ \frac{1}{\rho} A_{mn} & \text{if } \eta_m + \eta_n = \kappa + 1 \end{cases}$$

where the matrix $\frac{1}{\rho} A_{mn}$ is defined by equation (3.12) and its properties were obtained in proposition 3.4.

We recall that z^{r-1} is the coordinate of the lowest weight root vector $a = X_\kappa^{r-1}$. We denote Ξ_I^i the value $\langle X_I^i | X_I^i \rangle$ and set

$$[a, X_I^i] = \sum_j \Delta_I^{ij} X_{I-\kappa}^j.$$

The fact that $z^{r-1}(x)$ is the only invariant which depends on $q_{r-1}^\kappa(x)$ implies that the invariant $z^{r-1}(x)$ will appear in the expression of $\{z^i(x), z^j(y)\}^{\tilde{Q}}$ only if, when using the Leibnitz rule, we encounter terms of the original Poisson bracket $\{.,.\}$ depend explicitly on $q_{r-1}^\kappa(x)$. The later appear as a result of the following “brackets”

$$[q_j^{\kappa-I}(x), q_i^I(y)] := q_{r-1}^\kappa(x) \frac{\Delta_I^{ij}}{\Xi_I^i} \delta(x-y). \quad (4.17)$$

Hence the dependence of $\{z^i(x), z^j(y)\}^{\tilde{Q}}$ on $z^{r-1}(x)$ can be evaluated by imposing the Leibnitz rule on the “brackets” above. We get

$$\begin{aligned} [z^m(x), z^n(y)] &= \sum_{i,I;j} \sum_{l,h} \frac{\Delta_I^{ij}}{\Xi_I^i} \frac{\partial z^m(x)}{\partial (q_j^{\kappa-I})^{(l)}} \partial_x^l \left(\frac{\partial z^n(y)}{\partial (q_i^I)^{(h)}} \partial_y^h (q_{r-1}^\kappa(x) \delta(x-y)) \right) \\ &= \sum_{i,I;j} \sum_{l,h,\alpha,\beta} (-1)^h \binom{h}{\alpha} \binom{l}{\beta} \frac{\Delta_I^{ij}}{\Xi_I^i} \frac{\partial z^m(x)}{\partial (q_j^{\kappa-I})^{(l)}} (q_{r-1}^\kappa(x) \left(\frac{\partial z^n(x)}{\partial (q_i^I)^{(h)}} \right)^{(\alpha)})^{(\beta)} \delta^{(h+l-\alpha-\beta)}(x-y). \end{aligned}$$

Here we omitted the ranges of the indices since no confusion can arise. We observe that the coefficient of $\delta'(x-y)$ of this expression which contributes to the value of $g^{mn}(z)$ is given by

$$\mathbb{B}(z^m, z^n) = \sum_{i,I,J} \sum_{h,l} (-1)^h (l+h) \frac{\Delta_I^{ij}}{\Xi_I^i} q_{r-1}^\kappa(x) \frac{\partial z^m(x)}{\partial (q_j^{\kappa-I})^{(l)}} \left(\frac{\partial z^n(x)}{\partial (q_i^I)^{(h)}} \right)^{h+l-1} \quad (4.18)$$

Obviously, We get $g_1^{ij}(z)$ from the expression

$$\mathbb{A}(z^m, z^n) = \partial_{q_{r-1}^\kappa} \mathbb{B} = \sum_{i,I,J} \sum_{h,l} (-1)^h (l+h) \frac{\Delta_I^{ij}}{\Xi_I^i} \frac{\partial z^m(x)}{\partial (q_j^{\kappa-I})^{(l)}} \left(\frac{\partial z^n(x)}{\partial (q_i^I)^{(h)}} \right)^{h+l-1}. \quad (4.19)$$

Lemma 4.4. *The matrix $\mathbb{A}(z^m, z^n)$ is lower antidiagonal with respect to $Wt(e)$ and the antidiagonal entries are constants. In other word $\mathbb{A}(z^m, z^n)$ is constant if $\eta_m + \eta_n \leq \kappa + 1$ and equals zero if $\eta_m + \eta_n < \kappa + 1$.*

Proof. We note that if $z^m(x)$ and $z^n(x)$ are quasihomogenous of degree $2\eta_m + 2$ and $2\eta_n + 2$, respectively, then $\mathbb{A}(z^m, z^n)$ will be quasihomogenous of degree

$$2\eta_m + 2 + 2\eta_n + 2 - (2\kappa + 2) - 4 = 2\eta_m + 2\eta_n - 2\kappa - 2.$$

The proof is complete. \square

Proposition 4.5. *The minor matrix g_1^{mn} , $m, n = 1, \dots, r$ is nondegenerate and its determinant is equal to the determinant of the matrix $\frac{1}{\rho} A_{mn}$.*

Proof. We observe that, from our choice of coordinates, the minor matrix g_1^{mn} will be lower antidiagonal with respect to the set $Et(e)$. Hence, from the second part of proposition 3.4 we need only to prove that $\mathbb{A}(z^n, z^m)$ with $\eta_n + \eta_m = \kappa + 1$ is nonzero constant. In this case z^m and z^n are quasihomogenous of degree $2\eta_m + 2$ and $2\kappa - 2\eta_m + 4$, respectively. The expression

$$\mathbb{A}(z^n, z^m) = \sum_{i,I,J} \sum_{h,l} (-1)^h (l+h) \frac{\Delta_I^{ij}}{\Xi_I^i} \frac{\partial z^n(x)}{\partial (q_j^{\kappa-I})^{(l)}} \left(\frac{\partial z^m(x)}{\partial (q_i^I)^{(h)}} \right)^{h+l-1} \quad (4.20)$$

gives the constraints

$$\begin{aligned} 2I + 2 &\leq 2\eta_m + 2 \\ 2\kappa - 2I + 2 &\leq 2\kappa - 2\eta_m + 4 \end{aligned} \quad (4.21)$$

which implies

$$\eta_m - 1 \leq I \leq \eta_m$$

Therefore the only possible values for the index I in the expression of $\mathbb{A}(z^n, z^m)$ that make sense are η_m and $\eta_m - 1$. Consider the partial summation of $\mathbb{A}(z^n, z^m)$ when $I = \eta_m$. The degree of $z^m(x)$ yields $h = 0$ and that $z^m(x)$ depends linearly on $q_i^{\eta_m}(x)$. But then equation (4.9) implies that i is fixed and equals to m . A similar argument on $z^n(x)$ we find that the

indices l and j are fixed and equal to 1 and n , respectively. Hence, the partial summation when $I = \eta_m$ gives the value

$$\frac{\Delta_{\eta_m}^{mn}}{\Xi_{\eta_m}^m} \frac{\partial z^n(x)}{\partial(q_n^{\kappa-\eta_m})^{(1)}} \frac{\partial z^m(x)}{\partial(q_m^{\eta_m})^{(0)}} = -\frac{\Delta_{\eta_m}^{mn}}{\Xi_{\eta_m}^m}.$$

We now turn to the partial summation of $\mathbb{A}(z^n, z^m)$ when $I = \eta_m - 1$. The possible values for h are 1 and 0. When $h = 0$ we get zero since l and h can only be zero. When $h = 1$ we get, similar to the above calculation, the value

$$(-1) \frac{\Delta_{\eta_m-1}^{mn}}{\Xi_I^i} \frac{\partial z^n(x)}{\partial(q_n^{\kappa-\eta_m})^{(0)}} \frac{\partial z^m(x)}{\partial(q_m^{\eta_m-1})^{(1)}} = \frac{\Delta_{\eta_m-1}^{mn}}{\Xi_{\eta_m-1}^m}.$$

Hence we end with the expression

$$\begin{aligned} \mathbb{A}(z^n, z^m) &= \frac{\Delta_{\eta_m-1}^{mn}}{\Xi_{\eta_m-1}^m} - \frac{\Delta_{\eta_m}^{mn}}{\Xi_{\eta_m}^m} \\ &= \frac{\langle [a, X_{\eta_m}^n] | X_{\eta_m-1}^m \rangle}{2\eta_m} + \frac{\langle [a, X_{\eta_m}^m] | X_{\eta_m-1}^n \rangle}{2\eta_n} = \frac{1}{\rho} A_{mn} \end{aligned}$$

where the last equality was obtained in proposition 3.5. Hence, the determinant of minor matrix g_1^{mn} , $m, n = 1, \dots, r$ equals to the determinant of $\frac{1}{\rho} A_{mn}$ which is nondegenerate. \square

5. DIRAC REDUCTION

In this section we extract information about $\{.,.\}^{\tilde{Q}}$ using the fact that the Poisson bracket $\{.,.\}^{\tilde{Q}}$ can be obtained by preforming the Dirac reduction on $\{.,.\}$.

Let \mathbf{n} denote the dimension of \mathfrak{g} . We use proposition 3.1 to fix a total order ξ_I , $I = 1, \dots, \mathbf{n}$ of the basis X_I^i such that:

- (1) The first $r+2$ are the lowest weight vectors in the following order

$$X_{-\eta_1}^1 < X_{-\eta_2}^2 < \dots < X_{-\eta_{r+2}}^{r+2}. \quad (5.1)$$

- (2) The matrix

$$\langle \xi_I | \xi_J \rangle, \quad I, J = 1, \dots, \mathbf{n} \quad (5.2)$$

is antidiagonal.

Let ξ^I be the dual basis of ξ_I under $\langle . | . \rangle$. We observe that if $\xi_I \in \mathfrak{g}_\mu$ then $\xi^I \in \mathfrak{g}_{-\mu}$. Let c_K^{IJ} denotes the structure constant of \mathfrak{g} under the dual basis and we set $\tilde{g}^{IJ} = \langle \xi^I | \xi^J \rangle$. We extend the coordinates $z^i(x)$ on \tilde{Q} to all $\mathfrak{L}(\mathfrak{g})$ by setting for $q(x) \in \mathfrak{L}(\mathfrak{g})$

$$z^I(q(x)) = \langle q(x) - e | \xi^I \rangle, \quad I = 1, \dots, \mathbf{n}. \quad (5.3)$$

Then we consider the following matrix differential operator

$$\mathbb{F}^{IJ} = \epsilon \tilde{g}^{IJ} \partial_x + \tilde{F}^{IJ}, \quad (5.4)$$

where,

$$\tilde{F}^{IJ} = \sum_K (c_K^{IJ} z^K(x)).$$

In this notations the Poisson bracket $\{.,.\}$ is given by

$$\{z^I(x), z^J(y)\} = \mathbb{F}^{IJ} \frac{1}{\epsilon} \delta(x-y). \quad (5.5)$$

For the rest of this section we consider three types of indices which have different ranges; capital letters $I, J, K, \dots = 1, \dots, \mathbf{n}$, small letters $i, j, k, \dots = 1, \dots, r+2$ and Greek letters $\alpha, \beta, \delta, \dots = r+3, \dots, \mathbf{n}$.

We observe that the matrix \tilde{F}^{IJ} define the finite Lie-Poisson structure on \mathfrak{g} . It is well known that the symplectic subspaces of this structure are the orbit spaces of \mathfrak{g} under the adjoint action and there are r global Casimirs [22]. Since the Slodowy slice $Q = e + \mathfrak{g}^f$ is transversal to the orbit of e , the minor matrix $\tilde{F}^{\alpha\beta}$ is nondegenerate. Let $\tilde{F}_{\alpha\beta}$ denote its inverse.

Proposition 5.1. ([9],[19]) *The Poisson bracket $\{.,.\}^{\tilde{Q}}$ can be obtained by performing the Dirac reduction on $\{.,.\}$ on \tilde{Q} .*

By proposition 2.5, the leading terms of $\{.,.\}^{\tilde{Q}}$ are given by

$$F^{ij}(z(x)) = (\tilde{F}^{ij} - \tilde{F}^{i\beta} \tilde{F}_{\beta\alpha} \tilde{F}^{\alpha j}) \quad (5.6)$$

$$g^{ij}(z(x)) = \tilde{g}^{ij} - \tilde{g}^{i\beta} \tilde{F}_{\beta\alpha} \tilde{F}^{\alpha j} + \tilde{F}^{i\lambda} \tilde{F}_{\beta\alpha} \tilde{g}^{\alpha\varphi} \tilde{F}_{\varphi\gamma} \tilde{F}^{\gamma j} - \tilde{F}^{i\beta} \tilde{F}_{\beta\alpha} \tilde{g}^{\alpha j}. \quad (5.7)$$

and

$$\Gamma_k^{ij}(z(x)) z_x^k = -(\tilde{g}^{i\beta} - \tilde{F}^{i\lambda} \tilde{F}_{\lambda\alpha} \tilde{g}^{\alpha\beta}) \partial_x (\tilde{F}_{\beta\varphi} \tilde{F}^{\varphi j}) \quad (5.8)$$

A consequence of this proposition is the following

Proposition 5.2. [19] *The Poisson bracket $\{.,.\}^{\tilde{Q}}$ has the following form*

$$\{z^1(x), z^1(y)\}^{\tilde{Q}} = \epsilon \delta'''(x-y) + 2z^1(x) \delta'(x-y) + z_x^1 \delta(x-y), \quad (5.9)$$

$$\{z^1(x), z^i(y)\}^{\tilde{Q}} = (\eta_i + 1) z^i(x) \delta'(x-y) + \eta_i z_x^i \delta(x-y),$$

$i = 1, \dots, r+2$.

Indeed, equations (5.9) are exactly the identities which define Virasoro density and classical W -algebras [20].

5.1. The quasihomogeneity condition. we want to study the quasihomogeneity of the entries $g_2^{ij}(z)$. We use the definition of the coordinates $z^I(x)$ and we assign degree $\mu_I + 2$ to $z^I(x)$ if $\xi^I \in \mathfrak{g}_{\mu_I}$. These degrees agree with those given in corollary 4.3. From the total order of the basis, it follows that if $z^I(x)$ has degree $\mu_I + 2$ then degree $z^{\mathbf{n}-I+1}(x)$ equals $-\mu_I + 2$. Further, since $[\mathfrak{g}_{\mu_I}, \mathfrak{g}_{\mu_J}] \subset \mathfrak{g}_{\mu_I + \mu_J}$, an entry $\tilde{F}^{IJ}(x)$ is quasihomogenous of degree $\mu_I + \mu_J + 2$.

Definition 5.3. in considering the degrees of the coordinates z^I , We say a matrix $B^{IJ}(z)$ with polynomial entries is quasihomogenous of degree n if each entry $B^{IJ}(z)$ is quasihomogenous of degree $\mu_I + \mu_J + n$.

Proposition 5.4. [5] *The matrix $\tilde{F}_{\beta\alpha}(z)$ restricted to \tilde{Q} is polynomial and quasihomogenous of degree -2 .*

Proposition 5.5. *The matrix $g^{ij}(z)$ is quasihomogenous of degree -4 while the matrix $F^{ij}(z)$ is quasihomogenous of degree -2 and the matrix $\Gamma_k^{ij}(z)$ is quasihomogenous of degree $-(2\eta_k + 2) - 4$.*

Proof. The statement about the quasihomogeneity of the matrix $F^{ij}(z)$ was proved in [5]. We will derive the quasihomogeneity of $g^{ij}(z)$. We know that the matrix \tilde{g}^{IJ} is constant antidiagonal. Hence we can write $\tilde{g}^{IJ} = C^I \delta_{\mathbf{n}-J+1}^I$ where C^I are nonzero constant. In particular, we have $\tilde{g}^{ij}(z) = 0$. Now consider the expression (5.7). Then for a fixed i we have

$$\tilde{g}^{i\beta} \tilde{F}_{\beta\alpha} \tilde{F}^{\alpha j} = C^i \tilde{F}_{\mathbf{n}-i+1,\alpha} \tilde{F}^{\alpha j}. \quad (5.10)$$

Hence, the left hand sight is quasihomogenous of degree

$$\mu_j + \mu_\alpha + 2 - \mu_\alpha - (-\mu_i) - 2 = \mu_j + \mu_i = 2\eta_i + 2\eta_j.$$

A similar argument shows that $\tilde{F}^{i\beta}\tilde{F}_{\beta\alpha}\tilde{g}^{\alpha j}$ is quasihomogenous of degree $2\eta_i + 2\eta_j$. Finally, the summation

$$\tilde{F}^{i\beta}\tilde{F}_{\beta\alpha}\tilde{g}^{\alpha\varphi}\tilde{F}_{\varphi\gamma}\tilde{F}^{\gamma j} = \sum_{\alpha} C^{\alpha}\tilde{F}^{i\beta}\tilde{F}_{\beta\alpha}\tilde{F}_{\mathbf{n}-\alpha+1,\gamma}\tilde{F}^{\gamma j}. \quad (5.11)$$

Then, it has the degree

$$\mu_i + \mu_\beta + 2 - \mu_\beta - \mu_\alpha - 2 - \mu_{\mathbf{n}-\alpha+1} - \mu_\gamma - 2 + \mu_\gamma + \mu_j + 2 = 2\eta_i + 2\eta_j.$$

This proves that $g^{ij}(z)$ is quasihomogenous of degree -4 . For the last statement in the proposition we observe that the formula for $\Gamma_{2;k}^{ij}(z(x))$ is given by

$$\Gamma_k^{ij}(z(x)) = -(\tilde{g}^{i\beta} - \tilde{F}^{i\lambda}\tilde{F}_{\lambda\alpha}\tilde{g}^{\alpha\beta})\partial_{z^k}(\tilde{F}_{\beta\varphi}\tilde{F}^{\varphi j}). \quad (5.12)$$

Hence the calculation of quasihomogeneity will be same as equations (5.10) and (5.11) with subtracting $2\eta_k + 2$. This complete the proof. \square

5.2. Subregular classical W -algebra. Let us consider the Poisson bracket $\{.,.\}^{\tilde{Q}}$ in Slodowy coordinates (t^1, \dots, t^n) and write

$$\begin{aligned} \{t^i(x), t^j(y)\}^{[-1]} &= F^{ij}(t(x))\delta(x-y) \\ \{t^i(x), t^j(y)\}^{[0]} &= g^{ij}(t(x))\delta'(x-y) + \Gamma_k^{ij}(t(x))t_x^k\delta(x-y) \end{aligned} \quad (5.13)$$

Proposition 5.6. *The minor matrix $\partial_{t^{r-1}}g_1^{mn}(t)$, $m, n = 1, \dots, r$ is nondegenerate and its determinant is equal to the determinant of the matrix $\frac{1}{\rho}A_{ij}$. In particular, the minor matrix g^{mn} , $m, n = 1, \dots, r$ is generically nondegenerate. Moreover, we have the same identities which defines classical W -algebras, i.e*

$$\begin{aligned} \{t^1(x), t^1(y)\}^{\tilde{Q}} &= \epsilon\delta'''(x-y) + 2t^1(x)\delta'(x-y) + t_x^1\delta(x-y) \\ \{t^1(x), t^i(y)\}^{\tilde{Q}} &= (\eta_i + 1)t^i(x)\delta'(x-y) + \eta_i t_x^i\delta(x-y). \end{aligned} \quad (5.14)$$

Proof. The nondegeneracy statements follows from the fact that the proof of proposition 4.5 depends only on the linear terms of the invariant differential polynomials z^i (see proposition 3.7). For the second part of the statement, we need only to show that

$$g^{1,n}(t) = (\eta_i + 1)t^i, \quad \Gamma_k^{1j}(t) = \eta_j\delta_k^j. \quad (5.15)$$

Note that, from proposition 5.2, we have

$$g^{1,n}(z) = (\eta_i + 1)z^i, \quad \Gamma_k^{1j}(z) = \eta_j\delta_k^j. \quad (5.16)$$

If we introduce the Euler vector field

$$E' := \sum_i (\eta_i + 1)z^i\partial_{z^i}. \quad (5.17)$$

Then the formula for change of coordinates gives

$$g^{1j}(t) = \partial_{z^a}t^1\partial_{z^b}t^j g_2^{ab}(z) = E'(t^j) = (\eta_j + 1)t^j. \quad (5.18)$$

Where the last equality comes from quasihomogeneity of the coordinates t^i . For $\Gamma_k^{1j}(z)$, the change of coordinates has the following formula

$$\Gamma_k^{ij}(t)dt^k = \left(\partial_{z^a}t^i\partial_{z^c}\partial_{z^b}t^j g_2^{ab}(z) + \partial_{z^a}t^i\partial_{z^b}t^j\Gamma_c^{ab}(z)\right)dz^c. \quad (5.19)$$

But then we get

$$\begin{aligned}\Gamma_k^{1j} dt^k &= \left(E'(\partial_{z^c} t^j) + \partial_{z^b} t^j \Gamma_c^{1b} \right) dz^c \\ &= \left((\eta_j - \eta_c) \partial_{z^c} t^j + \eta_c \partial_{z^c} t^j \right) dz^c = \eta_j \partial_{z^c} t^j dz^c = \eta_j dt^j\end{aligned}\quad (5.20)$$

□

The following theorem was proved in [5] using Slodowy coordinates

Theorem 5.7. *The matrix $F^{ij}(t)$ is a constant multiple of the matrix*

$$\begin{pmatrix} 0 & 0 \\ 0 & \Omega \end{pmatrix} \quad (5.21)$$

where Ω is a 3×3 matrix of the form

$$\begin{pmatrix} 0 & \frac{\partial t^0}{\partial t^{r+2}} & -\frac{\partial t^0}{\partial t^{r+1}} \\ -\frac{\partial t^0}{\partial t^{r+2}} & 0 & \frac{\partial t^0}{\partial t^r} \\ \frac{\partial t^0}{\partial t^{r+1}} & -\frac{\partial t^0}{\partial t^r} & 0 \end{pmatrix} \quad (5.22)$$

where t^0 is the restriction to Q of the invariant polynomial χ^0 defined after proposition 3.7.

Let $N \subset Q$ be the hypersurface of dimension r defined as follows

$$N = \left\{ t \in Q : \frac{\partial t^0}{\partial t^{r+2}} = \frac{\partial t^0}{\partial t^{r+1}} = 0 \right\} \quad (5.23)$$

From the quasihomogeneity of t^0 and the table in page 13, we observe that $\frac{\partial t^0}{\partial t^{r+2}}$ depends linearly on t^{r+2} and $\frac{\partial t^0}{\partial t^{r+1}}$ is a polynomial in t^{r+1} of degree $\iota = r - 2$ (resp. $\iota = 2$) if \mathfrak{g} is a Lie algebra of type D_r (resp. E_r). In particular, (t^1, \dots, t^r) is well defined coordinates on N .

Theorem 5.8. *The Dirac reduction of $\{.,.\}^{\tilde{Q}}$ to $\tilde{N} = \mathfrak{L}(N)$ is well defined and gives a local Poisson brackets $\{.,.\}^{\tilde{N}}$. The Poisson bracket $\{.,.\}^{\tilde{N}}$ is a classical W -algebra. It admits a dispersionless limit and the leading term is a nondegenerate Poisson bracket of hydrodynamic type.*

Proof. We observe that the minor matrix

$$\begin{pmatrix} 0 & \frac{\partial t^0}{\partial t^r} \\ -\frac{\partial t^0}{\partial t^r} & 0 \end{pmatrix} \quad (5.24)$$

of the matrix $F^{ij}(t)$ is nondegenerate. Hence, from theorem 2.5, it follows that the Dirac reduction of $\{.,.\}^{\tilde{Q}}$ to $\tilde{N} = \mathfrak{L}(N)$ is well defined and gives a local Poisson brackets $\{.,.\}^{\tilde{N}}$.

Let us write the reduced Poisson bracket on \tilde{N} in the form

$$\{t^m(x), t^n(y)\}^{\tilde{N}} = \sum_{k=-1}^{\infty} \epsilon^k \{t^m(x), t^n(y)\}_{\tilde{N}}^{[k]}$$

where

$$\begin{aligned}\{t^m(x), t^n(y)\}_{\tilde{N}}^{[-1]} &= \hat{F}^{mn}(t(x)) \delta(x - y) \\ \{t^m(x), t^n(y)\}_{\tilde{N}}^{[0]} &= \hat{g}^{mn}(t(x)) \delta'(x - y) + \hat{\Gamma}_k^{mn}(t(x)) t_x^k \delta(x - y).\end{aligned}\quad (5.25)$$

Then, it follows from corollary 2.6 that the entries $\hat{g}^{mn}(t)$ and $\hat{F}^{ij}(t)$ equal $g^{mn}(t)$ and $F^{mn}(t)$, respectively, where t^{r+1} and t^{r+2} are solutions of equations (5.23). From proposition 5.6, this implies that $\{.,.\}^{\tilde{N}}$ is a classical W -algebra. Moreover, from proposition

5.7, we have $\widehat{F}^{ij} = 0$. Hence $\{.,.\}^{\widetilde{N}}$ admits a dispersionless limit. Furthermore, proposition 5.6 implies that $\widehat{g}^{mn}(t(x))$ is generically nondegenerate. Hence, $\{t^m(x), t^n(y)\}_{\widetilde{N}}^{[0]}$ is nondegenerate Poisson brackets of hydrodynamics type. \square

In addition to the fact that $\{.,.\}^{\widetilde{N}}$ gives a Frobenius structure. The construction by considering the theory of opposite Cartan subalgebra implies that it is very associated to the Drinfeld-Sokolov hierarchy obtained in [6] that $\{.,.\}^{\widetilde{Q}}$. Therefore, we call it **subregular classical W-algebra**.

6. ALGEBRAIC FROBENIUS MANIFOLD

In this section we obtain the promised algebraic Frobenius structure.

Let us consider, using the Dubrovin-Novikov theorem 2.4, the contravariant metric $\widehat{g}^{mn}(t)$ on N and its Levi-Civita connection $\widehat{\Gamma}_k^{mn}(t)$. From proposition 5.5, these matrices are linear in t^{r-1} . Hence, lemma 2.1 implies that the matrices

$$\widehat{g}_2^{mn}(t) = \widehat{g}^{mn}(t), \quad \widehat{g}_1^{mn}(t) = \partial_{t^{r-1}} \widehat{g}_2^{mn}(t) \quad (6.1)$$

form a flat pencil of metrics on N .

Proposition 6.1. *There exist quasihomogenous polynomials coordinates of degrees $2\eta_i + 2$ in the form*

$$s^i = t^i + T^i(t^1, \dots, t^{i-1})$$

such that the matrix $\widehat{g}_1^{ij}(s)$ is constant antidiagonal. Furthermore, in this coordinates the metric $g_2^{ij}(s)$ and its Levi-Civita connection have the following entries

$$g_2^{1,n}(s) = (\eta_i + 1)s^i, \quad \Gamma_{2k}^{1j}(s) = \eta_j \delta_k^j \quad (6.2)$$

Proof. The proof of the first part of the proposition is given in [12] using the quasihomogeneity property of the matrix \widehat{g}^{mn} . The second part is obtained in the same manor as in proposition 5.2. \square

We assume without lost of generality that the coordinates t^i are the flat coordinates for \widehat{g}_1^{ij} .

Theorem 6.2. *The flat pencil of metrics given by $\widehat{g}_1^{mn}(t)$ and $\widehat{g}_2^{mn}(t)$ on the space N is regular quasihomogenous of degree $d = \frac{\kappa-1}{\kappa+1}$.*

Proof. In the notations of definition 2.2 we take $\tau = \frac{1}{\kappa+1}t^1$ then

$$\begin{aligned} E &= g_2^{ij} \partial_{t^j} \tau \partial_{t^i} = \frac{1}{\kappa+1} \sum_i (\eta_i + 1) t^i \partial_{t^i}, \\ e &= g_1^{ij} \partial_{t^j} \tau \partial_{t^i} = \partial_{t^{r-1}}. \end{aligned} \quad (6.3)$$

We see immediately that

$$[e, E] = e$$

The identity

$$\mathfrak{L}_e(\cdot, \cdot)_2 = (\cdot, \cdot)_1 \quad (6.4)$$

follows from the fact that $\partial_{t^{r-1}} = \partial_{z^{r-1}}$. Then

$$\mathfrak{L}_e(\cdot, \cdot)_1 = 0. \quad (6.5)$$

is a consequence from the quasihomogeneity of the matrix g_1^{ij} (see lemma 4.4). We also obtain from proposition 5.5 that

$$\mathfrak{L}_E(,)_2(dt^i, dt^j) = E(g_2^{ij}) - \frac{\eta_i + 1}{\kappa + 1} g_2^{ij} - \frac{\eta_j + 1}{\kappa + 1} g_2^{ij} = \frac{-2}{\kappa + 1} g_2^{ij}. \quad (6.6)$$

Hence,

$$\mathfrak{L}_E(,)_2 = (d - 1)(,)_2 \quad (6.7)$$

It remains to prove the regularity condition. But the (1,1)-tensor is nondegenerate since it has the entries

$$R_i^j = \frac{d - 1}{2} \delta_i^j + \nabla_{1_i} E^j = \frac{\eta_i}{\kappa + 1} \delta_i^j. \quad (6.8)$$

This complete the proof. \square

Now we have all the tools to prove the following

Theorem 6.3. *The space N has a natural structure of algebraic Frobenius manifold with charge $\frac{\kappa-1}{\kappa+1}$ and degrees $\frac{\eta_i+1}{\kappa+1}$, $i = 1, \dots, r$.*

Proof. It follows from theorem 6.2 and 2.3 that N has a Frobenius structure of degree $\frac{\kappa-1}{\kappa+1}$. This Frobenius structure is algebraic since in the coordinates t^i the potential \mathbb{F} is constructed using equations (2.9). Besides we have from theorem 5.8 the matrix \widehat{g}_2^{mn} depends on the nontrivial solutions of equations (5.23). \square

6.1. The algebraic Frobenius manifold of $D_4(a_1)$. We verify the procedure, outlined in this work, of constructing algebraic Frobenius manifold when \mathfrak{g} is of type D_4 . For this end we choose the realization of D_4 as a subalgebra of $gl_8(\mathbb{C})$ given in the appendix of [11]. In this case it is easy to obtain a representation of e which belongs to strictly lower diagonal matrices. In what follows we will denote by $\sigma_{i,j}$ the standard matrix defined by $(\sigma_{i,j})_{k,l} = \delta_{i,k} \delta_{j,l} \in gl_8(\mathbb{C})$. We fix the subregular nilpotent element e and the sl_2 -triple $\{e, h, f\}$ as follows

$$e = \sigma_{2,1} - \sigma_{3,1} + \sigma_{4,3} - \frac{\sigma_{5,2}}{2} + \sigma_{6,5} + \frac{\sigma_{7,4}}{2} + \sigma_{8,6} + \sigma_{8,7} \quad (6.9)$$

$$h = -4\sigma_{1,1} - 2\sigma_{2,2} - 2\sigma_{3,3} + 2\sigma_{6,6} + 2\sigma_{7,7} + 4\sigma_{8,8} \quad (6.10)$$

$$f = 2\sigma_{1,2} - 2\sigma_{1,3} - 2\sigma_{2,4} - 8\sigma_{2,5} + 4\sigma_{3,4} + 4\sigma_{3,5} + 4\sigma_{4,6} + 8\sigma_{4,7} + 4\sigma_{5,6} + 2\sigma_{5,7} + 2\sigma_{6,8} + 2\sigma_{7,8} \quad (6.11)$$

We observe that $Wt(e) = \{1, 3, 3, 1, 1, 3\}$ and $Et(e) = \{1, 3, 3, 1\}$. We construct a basis for \mathfrak{g} satisfy the hypotheses of proposition 3.1 from the formula

$$X_I^i = \frac{1}{(\eta_i + I)} \text{ad}^{\eta_i + I} e X_{-\eta_i}^i, \quad i = 1, \dots, 6. \quad (6.12)$$

where the lowest root vectors $X_{-\eta_i}^i$ are

$$\begin{aligned}
X_{-3}^2 &= 24\sqrt{3}\sigma_{3,8} - 24\sqrt{3}\sigma_{1,6} \\
X_{-3}^3 &= -24\sigma_{1,6} - 48\sigma_{1,7} - 48\sigma_{2,8} + 24\sigma_{3,8} \\
X_{-1}^4 &= -4\sqrt{\frac{3}{5}}\sigma_{1,2} - 2\sqrt{\frac{3}{5}}\sigma_{1,3} + 2\sqrt{\frac{3}{5}}\sigma_{2,4} + 2\sqrt{\frac{3}{5}}\sigma_{3,4} - 12\sqrt{\frac{3}{5}}\sigma_{3,5} \\
&\quad - 12\sqrt{\frac{3}{5}}\sigma_{4,6} + 2\sqrt{\frac{3}{5}}\sigma_{5,6} - 2\sqrt{\frac{3}{5}}\sigma_{5,7} + 2\sqrt{\frac{3}{5}}\sigma_{6,8} - 4\sqrt{\frac{3}{5}}\sigma_{7,8} \\
X_{-1}^5 &= -\frac{8\sigma_{1,2}}{\sqrt{5}} - 2\sqrt{5}\sigma_{1,3} - 2\sqrt{5}\sigma_{2,4} + \frac{8\sigma_{2,5}}{\sqrt{5}} + \frac{2\sigma_{3,4}}{\sqrt{5}} - \frac{4\sigma_{3,5}}{\sqrt{5}} \\
&\quad - \frac{4\sigma_{4,6}}{\sqrt{5}} - \frac{8\sigma_{4,7}}{\sqrt{5}} + \frac{2\sigma_{5,6}}{\sqrt{5}} + 2\sqrt{5}\sigma_{5,7} + 2\sqrt{5}\sigma_{6,8} - \frac{8\sigma_{7,8}}{\sqrt{5}}. \\
X_{-2}^6 &= -4\sqrt{3}\sigma_{1,4} + 8\sqrt{3}\sigma_{1,5} + 8\sqrt{3}\sigma_{2,6} + 8\sqrt{3}\sigma_{3,7} + 8\sqrt{3}\sigma_{4,8} - 4\sqrt{3}\sigma_{5,8}.
\end{aligned} \tag{6.13}$$

The opposite Cartan subalgebra \mathfrak{h}' have the following normalized basis

$$\begin{aligned}
y_1 &= e + X_{-3}^4 \\
y_2 &= -X_3^2 - \frac{3}{\sqrt{5}}X_{-1}^4 - \frac{1}{5\sqrt{3}}X_{-1}^3 + \frac{1}{2}X_{-1}^6 \\
y_3 &= -X_3^3 + 3X_{-1}^1 - \frac{1}{5\sqrt{3}}X_{-1}^2 + \frac{3}{\sqrt{5}}X_{-1}^5 \\
y_4 &= -X_1^4 - \frac{1}{\sqrt{5}}X_{-3}^2.
\end{aligned} \tag{6.14}$$

The matrix of the restriction of $\langle \cdot | \cdot \rangle$ to \mathfrak{h}' under the order $\{y_1, y_4, y_2, y_3\}$ equals

$$\begin{pmatrix} 0 & 0 & 0 & 4 \\ 0 & 0 & -\frac{4}{\sqrt{5}} & 0 \\ 0 & -\frac{4}{\sqrt{5}} & 0 & 0 \\ 4 & 0 & 0 & 0 \end{pmatrix} \tag{6.15}$$

We write an element z in Slodowy slice Q in the form

$$z = z_1X_{-1}^1 + z_2X_{-3}^3 + z_3X_{-3}^4 + z_4X_{-1}^2 + z_5X_{-1}^5 + z_6X_{-2}^6 + e. \tag{6.16}$$

Here we lower the index for convenience. Then the restriction of the invariant polynomials to Q can be found from the coefficients of the indeterminate P in the equation $\det(z - P)$. After normalization we get the following Slodowy coordinates on Q

$$\begin{aligned}
t_1 &= z_1 \\
t_2 &= z_2 - \frac{1}{2}\sqrt{3}z_1^2 + \frac{4z_5z_1}{\sqrt{15}} + \frac{7z_4^2}{5\sqrt{3}} - \frac{7z_5^2}{5\sqrt{3}} \\
t_3 &= z_3 - \frac{3z_1^2}{2} - \frac{4z_4z_1}{\sqrt{15}} - \frac{14z_4z_5}{5\sqrt{3}} \\
t_i &= z_i, \quad i = 4, 5, 6.
\end{aligned} \tag{6.17}$$

We take the following as the restriction to Q of a highest degree invariant polynomial

$$\begin{aligned}
t_0 &= 20t_1^3 + 18\sqrt{15}t_4t_1^2 - 18\sqrt{5}t_5^2t_1^2 + 60t_4^2t_1 + 60t_5^2t_1 + 6\sqrt{3}t_2t_1 \\
&\quad + 18t_3t_1 - 20\sqrt{5}t_5^3 - 27t_6^2 + 12\sqrt{15}t_3t_4 + 60\sqrt{5}t_4^2t_5 - 12\sqrt{15}t_2t_5
\end{aligned} \tag{6.18}$$

Note that in the case $t_i = 0$, $i = 1, 2, 3$ we get the equation

$$f(t_4, t_5, t_6) = -20\sqrt{5}t_5^3 + 60\sqrt{5}t_4^2t_5 - 27t_6^2 \quad (6.19)$$

which define a simple hypersurface singularity of type D_4 .

It follows that the leading term of the classical W -algebra on \tilde{Q} is given by

$$F^{ij}(t) = 75 \begin{pmatrix} 0 & 0 \\ 0 & \Omega \end{pmatrix} \quad (6.20)$$

where Ω is a 3×3 matrix of theorem 5.7. The hypersurface $N \subset Q$ is defined by the equations

$$\begin{aligned} \frac{\partial t_0}{\partial t_6} &= 270t_6 = 0 \\ \frac{\partial t_0}{\partial t_5} &= -5 \left(-18\sqrt{5}t_1^2 + 120t_5t_1 + 60\sqrt{5}t_4^2 - 60\sqrt{5}t_5^2 - 12\sqrt{15}t_2 \right) = 0 \end{aligned} \quad (6.21)$$

We choose the flat coordinates

$$\begin{aligned} s_1 &= t_1 \\ s_2 &= t_2 + \frac{\sqrt{3}}{4}t_1^2 - \frac{5\sqrt{3}4}{t}^2 \\ s_3 &= t_3 + \frac{3}{2}t_1^2 + \frac{\sqrt{15}}{2}t_4t_1 \\ s_4 &= t_4 \end{aligned} \quad (6.22)$$

Then the potential \mathbb{F} of the Frobenius structure reads

$$\begin{aligned} \mathbb{F} &= \frac{Z}{180} \left(-\sqrt{5}s_1^4 - 10\sqrt{5}s_4^2s_1^2 + 8\sqrt{15}s_2s_1^2 - 25\sqrt{5}s_4^4 - 48\sqrt{5}s_2^2 + 40\sqrt{15}s_2s_4^2 \right) \\ &+ \frac{1}{2880} \left(35s_1^5 + 510s_4^2s_1^3 - 48\sqrt{3}s_2s_1^3 + 775s_4^4s_1 + 360s_3^2s_1 \right. \\ &\quad \left. - 720\sqrt{5}s_2s_3s_4 + 1128s_2^2s_1 - 1840\sqrt{3}s_2s_4^2s_1 \right) \end{aligned} \quad (6.23)$$

where Z is a solution of the quadratic equation

$$Z^2 - \frac{2}{\sqrt{5}}s_1Z + \frac{3}{20}s_1^2 - \frac{1}{4}s_4^2 + \frac{\sqrt{3}}{5}s_2 = 0. \quad (6.24)$$

It is straightforward to check validity of the WDVV equations for this potential. The identity vector field is $\frac{\partial}{\partial s_3}$ and the quasihomogeneity reads

$$\frac{1}{2}s_1 \frac{\partial \mathbb{F}}{\partial s_1} + s_2 \frac{\partial \mathbb{F}}{\partial s_2} + s_3 \frac{\partial \mathbb{F}}{\partial s_3} + \frac{1}{2}s_4 \frac{\partial \mathbb{F}}{\partial s_4} = \frac{5}{2}\mathbb{F}. \quad (6.25)$$

7. CONCLUSIONS AND REMARKS

In this work we obtained infinite number of examples of algebraic Frobenius manifolds. These examples correspond to the regular quasi-Coxeter conjugacy classes $D_r(a_1)$ where r is even and $E_r(a_1)$. One of the tools we use is the structure of opposite Cartan subalgebra which relate the subregular nilpotent orbit to the conjugacy class. The structure of opposite Cartan subalgebra exists only for regular conjugacy class. But taking the subregular nilpotent orbit in the Lie algebra D_5 we obtain algebraic Frobenius manifold related to the nonregular quasi-Coxeter conjugacy class $D_5(a_1)$. This implies that the existence of

algebraic Frobenius manifold is a far deeper than the notion of opposite Cartan subalgebra. This fact will be the frame work of our future research. Our next step is to develop a method to uniform the construction of all algebraic Frobenius manifolds that could be obtained from quasi-Coxeter conjugacy classes in Weyl groups.

In this work we give, for the first time, a geometric realization of algebraic Frobenius manifolds. The examples obtained are certain hypersurfaces in the total spaces of semi-universal deformations of simple hypersurface singularities. We hope this will rich the relation between Frobenius manifolds and singularity theory. Which one of its main contributions is the existence of polynomial Frobenius structures on the universal unfolding of simple hypersurface singularities. For the definition and deference between semiuniversal and unfolding see chapter 2 section 1 of [18].

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